Step 2 We assume that $P(k)$ is true. Thus our induction hypothesis is

$$(a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k$$

We use this to show that $P(k+1)$ is true.

$$\begin{aligned}(a+b)^{k+1} &= (a+b)[(a+b)^k]\\
&= (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] && \text{Induction hypothesis}\\
&= a\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right]\\
&\quad + b\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] && \text{Distributive Property}\\
&= \binom{k}{0}a^{k+1} + \binom{k}{1}a^{k}b + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k}ab^k\\
&\quad + \binom{k}{0}a^{k}b + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k-1}ab^{k} + \binom{k}{k}b^{k+1} && \text{Distributive Property}\\
&= \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^kb + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2\\
&\quad + \cdots + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}b^{k+1} && \text{Group like terms}\end{aligned}$$

Using the key property of the binomial coefficients, we can write each of the expressions in square brackets as a single binomial coefficient. Also, writing the first and last coefficients as $\binom{k+1}{0}$ and $\binom{k+1}{k+1}$ (these are equal to 1 by Exercise 50) gives

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^kb + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1}$$

But this last equation is precisely $P(k+1)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that the theorem is true for all natural numbers n. ■

12.6 EXERCISES

CONCEPTS

1. An algebraic expression of the form $a + b$, which consists of a sum of two terms, is called a ________.

2. We can find the coefficients in the expansion of $(a+b)^n$ from the nth row of ________ triangle. So
$(a+b)^4 = \blacksquare a^4 + \blacksquare a^3b + \blacksquare a^2b^2 + \blacksquare ab^3 + \blacksquare b^4$

3. The binomial coefficients can be calculated directly by using the formula $\binom{n}{k}$ = ________. So $\binom{4}{3}$ = ________.

4. To expand $(a+b)^n$, we can use the ________ Theorem. Using this theorem, we find
$(a+b)^4 =$
$\binom{\blacksquare}{\blacksquare}a^4 + \binom{\blacksquare}{\blacksquare}a^3b + \binom{\blacksquare}{\blacksquare}a^2b^2 + \binom{\blacksquare}{\blacksquare}ab^3 + \binom{\blacksquare}{\blacksquare}b^4$

SKILLS

5–16 ■ Use Pascal's triangle to expand the expression.

5. $(x+y)^6$ **6.** $(2x+1)^4$ **7.** $\left(x+\frac{1}{x}\right)^4$

8. $(x-y)^5$ **9.** $(x-1)^5$ **10.** $(\sqrt{a}+\sqrt{b})^6$

11. $(x^2y-1)^5$ **12.** $(1+\sqrt{2})^6$ **13.** $(2x-3y)^3$

14. $(1+x^3)^3$ **15.** $\left(\frac{1}{x}-\sqrt{x}\right)^5$ **16.** $\left(2+\frac{x}{2}\right)^5$

17–24 ■ Evaluate the expression.

17. $\binom{6}{4}$ **18.** $\binom{8}{3}$ **19.** $\binom{100}{98}$

20. $\binom{10}{5}$ **21.** $\binom{3}{1}\binom{4}{2}$ **22.** $\binom{5}{2}\binom{5}{3}$

23. $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$

24. $\binom{5}{0}-\binom{5}{1}+\binom{5}{2}-\binom{5}{3}+\binom{5}{4}-\binom{5}{5}$

25–28 ■ Use the Binomial Theorem to expand the expression.

25. $(x+2y)^4$ **26.** $(1-x)^5$

27. $\left(1+\frac{1}{x}\right)^6$ **28.** $(2A+B^2)^4$

29. Find the first three terms in the expansion of $(x+2y)^{20}$.

30. Find the first four terms in the expansion of $(x^{1/2}+1)^{30}$.

31. Find the last two terms in the expansion of $(a^{2/3}+a^{1/3})^{25}$.

32. Find the first three terms in the expansion of

$$\left(x+\frac{1}{x}\right)^{40}$$

33. Find the middle term in the expansion of $(x^2+1)^{18}$.

34. Find the fifth term in the expansion of $(ab-1)^{20}$.

35. Find the 24th term in the expansion of $(a+b)^{25}$.

36. Find the 28th term in the expansion of $(A-B)^{30}$.

37. Find the 100th term in the expansion of $(1+y)^{100}$.

38. Find the second term in the expansion of

$$\left(x^2-\frac{1}{x}\right)^{25}$$

39. Find the term containing x^4 in the expansion of $(x+2y)^{10}$.

40. Find the term containing y^3 in the expansion of $(\sqrt{2}+y)^{12}$.

41. Find the term containing b^8 in the expansion of $(a+b^2)^{12}$.

42. Find the term that does not contain x in the expansion of

$$\left(8x+\frac{1}{2x}\right)^8$$

43–46 ■ Factor using the Binomial Theorem.

43. $x^4+4x^3y+6x^2y^2+4xy^3+y^4$

44. $(x-1)^5+5(x-1)^4+10(x-1)^3$
$+10(x-1)^2+5(x-1)+1$

45. $8a^3+12a^2b+6ab^2+b^3$

46. $x^8+4x^6y+6x^4y^2+4x^2y^3+y^4$

47–52 ■ Simplify using the Binomial Theorem.

47. $\dfrac{(x+h)^3-x^3}{h}$ **48.** $\dfrac{(x+h)^4-x^4}{h}$

49. Show that $(1.01)^{100}>2$. [*Hint:* Note that $(1.01)^{100}=(1+0.01)^{100}$, and use the Binomial Theorem to show that the sum of the first two terms of the expansion is greater than 2.]

50. Show that $\binom{n}{0}=1$ and $\binom{n}{n}=1$.

51. Show that $\binom{n}{1}=\binom{n}{n-1}=n$.

52. Show that $\binom{n}{r}=\binom{n}{n-r}$ for $0\le r\le n$.

53. In this exercise we prove the identity

$$\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$$

(a) Write the left-hand side of this equation as the sum of two fractions.
(b) Show that a common denominator of the expression that you found in part (a) is $r!(n-r+1)!$.
(c) Add the two fractions using the common denominator in part (b), simplify the numerator, and note that the resulting expression is equal to the right-hand side of the equation.

54. Prove that $\binom{n}{r}$ is an integer for all n and for $0\le r\le n$. [*Suggestion*: Use induction to show that the statement is true for all n, and use Exercise 53 for the induction step.]

APPLICATIONS

55. Difference in Volumes of Cubes The volume of a cube of side x inches is given by $V(x)=x^3$, so the volume of a cube of side $x+2$ inches is given by $V(x+2)=(x+2)^3$. Use the Binomial Theorem to show that the difference in volume between the larger and smaller cubes is $6x^2+12x+8$ cubic inches.

56. Probability of Hitting a Target The probability that an archer hits the target is $p=0.9$, so the probability that he misses the target is $q=0.1$. It is known that in this situation the probability that the archer hits the target exactly r times in n attempts is given by the term containing p^r in the binomial expansion of $(p+q)^n$. Find the probability that the archer hits the target exactly three times in five attempts.

DISCOVERY ■ DISCUSSION ■ WRITING

57. Powers of Factorials Which is larger, $(100!)^{101}$ or $(101!)^{100}$? [*Hint:* Try factoring the expressions. Do they have any common factors?]

58. Sums of Binomial Coefficients Add each of the first five rows of Pascal's triangle, as indicated. Do you see a pattern?

$$1 + 1 = \boxed{?}$$
$$1 + 2 + 1 = \boxed{?}$$
$$1 + 3 + 3 + 1 = \boxed{?}$$
$$1 + 4 + 6 + 4 + 1 = \boxed{?}$$
$$1 + 5 + 10 + 10 + 5 + 1 = \boxed{?}$$

On the basis of the pattern you have found, find the sum of the nth row:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Prove your result by expanding $(1 + 1)^n$ using the Binomial Theorem.

59. Alternating Sums of Binomial Coefficients Find the sum

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$$

by finding a pattern as in Exercise 58. Prove your result by expanding $(1 - 1)^n$ using the Binomial Theorem.

CHAPTER 12 | REVIEW

CONCEPT CHECK

1. **(a)** What is a sequence?
   (b) What is an arithmetic sequence? Write an expression for the nth term of an arithmetic sequence.
   (c) What is a geometric sequence? Write an expression for the nth term of a geometric sequence.

2. **(a)** What is a recursively defined sequence?
   (b) What is the Fibonacci sequence?

3. **(a)** What is meant by the partial sums of a sequence?
   (b) If an arithmetic sequence has first term a and common difference d, write an expression for the sum of its first n terms.
   (c) If a geometric sequence has first term a and common ratio r, write an expression for the sum of its first n terms.
   (d) Write an expression for the sum of an infinite geometric series with first term a and common ratio r. For what values of r is your formula valid?

4. **(a)** Write the sum $\sum_{k=1}^{n} a_k$ without using sigma notation.
   (b) Write $b_1 + b_2 + b_3 + \cdots + b_n$ using sigma notation.

5. Write an expression for the amount A_f of an annuity consisting of n regular equal payments of size R with interest rate i per time period.

6. State the Principle of Mathematical Induction.

7. Write the first five rows of Pascal's triangle. How are the entries related to each other?

8. **(a)** What does the symbol $n!$ mean?
   (b) Write an expression for the binomial coefficient $\binom{n}{r}$.
   (c) State the Binomial Theorem.
   (d) Write the term that contains a^r in the expansion of $(a + b)^n$.

EXERCISES

1–6 ■ Find the first four terms as well as the tenth term of the sequence with the given nth term.

1. $a_n = \dfrac{n^2}{n + 1}$

2. $a_n = (-1)^n\dfrac{2^n}{n}$

3. $a_n = \dfrac{(-1)^n + 1}{n^3}$

4. $a_n = \dfrac{n(n + 1)}{2}$

5. $a_n = \dfrac{(2n)!}{2^n n!}$

6. $a_n = \dbinom{n + 1}{2}$

7–10 ■ A sequence is defined recursively. Find the first seven terms of the sequence.

7. $a_n = a_{n-1} + 2n - 1, \quad a_1 = 1$

8. $a_n = \dfrac{a_{n-1}}{n}, \quad a_1 = 1$

9. $a_n = a_{n-1} + 2a_{n-2}, \quad a_1 = 1, a_2 = 3$

10. $a_n = \sqrt{3a_{n-1}}, \quad a_1 = \sqrt{3}$

SECTION 12.6 ■ PAGE 827

1. binomial **2.** Pascal's; 1, 4, 6, 4, 1

3. $\frac{n!}{k!(n-k)!}$; $\frac{4!}{3!(4-3)!} = 4$

4. Binomial; $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$

5. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

7. $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

9. $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$

11. $x^{10}y^5 - 5x^8y^4 + 10x^6y^3 - 10x^4y^2 + 5x^2y - 1$

13. $8x^3 - 36x^2y + 54xy^2 - 27y^3$

15. $\frac{1}{x^5} - \frac{5}{x^{7/2}} + \frac{10}{x^2} - \frac{10}{x^{1/2}} + 5x - x^{5/2}$

17. 15 **19.** 4950 **21.** 18 **23.** 32

25. $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$

27. $1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}$

29. $x^{20} + 40x^{19}y + 760x^{18}y^2$ **31.** $25a^{26/3} + a^{25/3}$

33. $48{,}620x^{18}$ **35.** $300a^2b^{23}$ **37.** $100y^{99}$ **39.** $13{,}440x^4y^6$

41. $495a^8b^8$ **43.** $(x + y)^4$ **45.** $(2a + b)^3$ **47.** $3x^2 + 3xh + h^2$