SOLUTION The polynomials

$$P_0(x) = x^3 \qquad P_1(x) = x^3 - x^2$$
$$P_2(x) = x^3 - 2x^2 \qquad P_3(x) = x^3 - 3x^2$$

are graphed in Figure 14. We see that increasing the value of c causes the graph to develop an increasingly deep "valley" to the right of the y-axis, creating a local maximum at the origin and a local minimum at a point in Quadrant IV. This local minimum moves lower and farther to the right as c increases. To see why this happens, factor $P(x) = x^2(x - c)$. The polynomial P has zeros at 0 and c, and the larger c gets, the farther to the right the minimum between 0 and c will be.

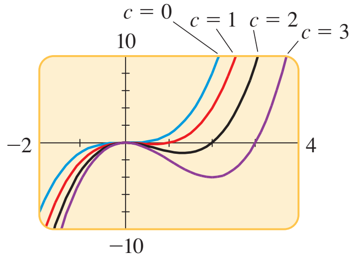


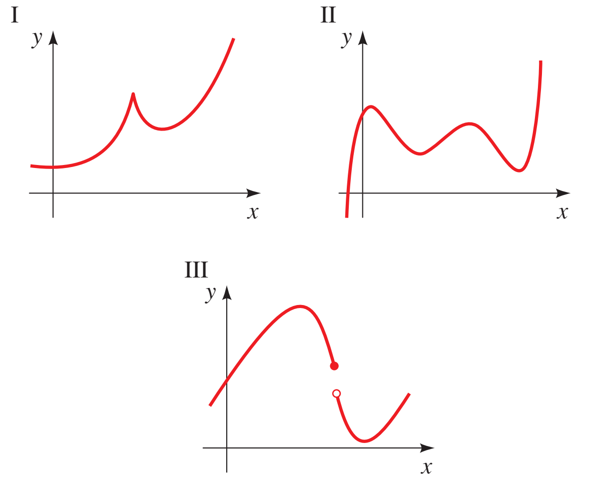
FIGURE 14 A family of polynomials $P(x) = x^3 - cx^2$

NOW TRY EXERCISE 71

3.2 EXERCISES

CONCEPTS

1. Only one of the following graphs could be the graph of a polynomial function. Which one? Why are the others not graphs of polynomials?



2. Every polynomial has one of the following behaviors:

(i) $y \to \infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$
(ii) $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$
(iii) $y \to -\infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$
(iv) $y \to -\infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$

For each polynomial, choose the appropriate description of its end behavior from the list above.

(a) $y = x^3 - 8x^2 + 2x - 15$: end behavior ________.

(b) $y = -2x^4 + 12x + 100$: end behavior ________.

3. If c is a zero of the polynomial P, which of the following statements must be true?

(a) $P(c) = 0$. **(b)** $P(0) = c$.
(c) $x - c$ is a factor of $P(x)$.
(d) c is the y-intercept of the graph of P.

4. Which of the following statements couldn't possibly be true about the polynomial function P?

(a) P has degree 3, two local maxima, and two local minima.
(b) P has degree 3 and no local maxima or minima.
(c) P has degree 4, one local maximum, and no local minima.

SKILLS

5–8 ■ Sketch the graph of each function by transforming the graph of an appropriate function of the form $y = x^n$ from Figure 2. Indicate all x- and y-intercepts on each graph.

5. **(a)** $P(x) = x^2 - 4$ **(b)** $Q(x) = (x - 4)^2$
(c) $R(x) = 2x^2 - 2$ **(d)** $S(x) = 2(x - 2)^2$

6. **(a)** $P(x) = x^4 - 16$ **(b)** $Q(x) = (x + 2)^4$
(c) $R(x) = (x + 2)^4 - 16$ **(d)** $S(x) = -2(x + 2)^4$

7. **(a)** $P(x) = x^3 - 8$ **(b)** $Q(x) = -x^3 + 27$
(c) $R(x) = -(x + 2)^3$ **(d)** $S(x) = \frac{1}{2}(x - 1)^3 + 4$

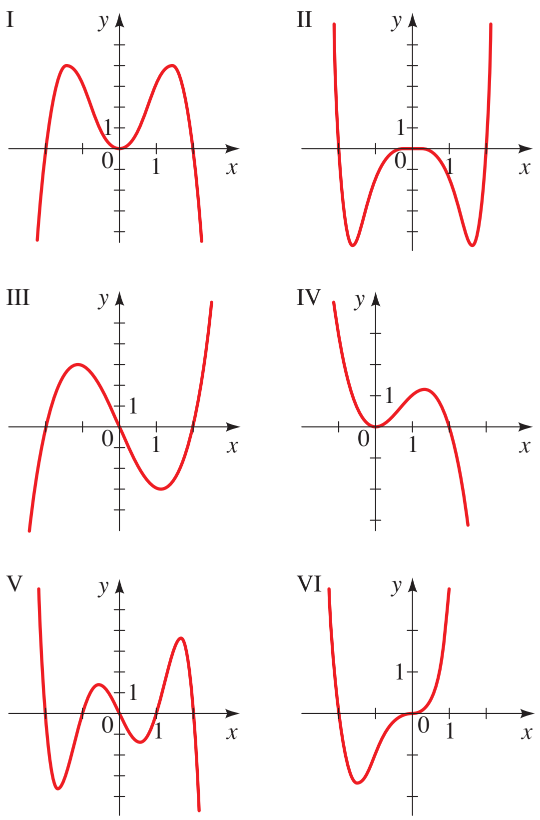
8. **(a)** $P(x) = (x + 3)^5$ **(b)** $Q(x) = 2(x + 3)^5 - 64$
(c) $R(x) = -\frac{1}{2}(x - 2)^5$ **(d)** $S(x) = -\frac{1}{2}(x - 2)^5 + 16$

9–14 ■ Match the polynomial function with one of the graphs I–VI on the next page. Give reasons for your choice.

9. $P(x) = x(x^2 - 4)$ **10.** $Q(x) = -x^2(x^2 - 4)$

11. $R(x) = -x^5 + 5x^3 - 4x$ **12.** $S(x) = \frac{1}{2}x^6 - 2x^4$

13. $T(x) = x^4 + 2x^3$ **14.** $U(x) = -x^3 + 2x^2$



15–26 ■ Sketch the graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior.

15. $P(x) = (x - 1)(x + 2)$

16. $P(x) = (x - 1)(x + 1)(x - 2)$

17. $P(x) = x(x - 3)(x + 2)$

18. $P(x) = (2x - 1)(x + 1)(x + 3)$

19. $P(x) = (x - 3)(x + 2)(3x - 2)$

20. $P(x) = \frac{1}{5}x(x - 5)^2$

21. $P(x) = (x - 1)^2(x - 3)$ **22.** $P(x) = \frac{1}{4}(x + 1)^3(x - 3)$

23. $P(x) = \frac{1}{12}(x + 2)^2(x - 3)^2$ **24.** $P(x) = (x - 1)^2(x + 2)^3$

25. $P(x) = x^3(x + 2)(x - 3)^2$ **26.** $P(x) = (x - 3)^2(x + 1)^2$

27–40 ■ Factor the polynomial and use the factored form to find the zeros. Then sketch the graph.

27. $P(x) = x^3 - x^2 - 6x$ **28.** $P(x) = x^3 + 2x^2 - 8x$

29. $P(x) = -x^3 + x^2 + 12x$ **30.** $P(x) = -2x^3 - x^2 + x$

31. $P(x) = x^4 - 3x^3 + 2x^2$ **32.** $P(x) = x^5 - 9x^3$

33. $P(x) = x^3 + x^2 - x - 1$ **34.** $P(x) = x^3 + 3x^2 - 4x - 12$

35. $P(x) = 2x^3 - x^2 - 18x + 9$

36. $P(x) = \frac{1}{8}(2x^4 + 3x^3 - 16x - 24)^2$

37. $P(x) = x^4 - 2x^3 - 8x + 16$

38. $P(x) = x^4 - 2x^3 + 8x - 16$

39. $P(x) = x^4 - 3x^2 - 4$ **40.** $P(x) = x^6 - 2x^3 + 1$

41–46 ■ Determine the end behavior of P. Compare the graphs of P and Q in large and small viewing rectangles, as in Example 3(b).

41. $P(x) = 3x^3 - x^2 + 5x + 1$; $Q(x) = 3x^3$

42. $P(x) = -\frac{1}{8}x^3 + \frac{1}{4}x^2 + 12x$; $Q(x) = -\frac{1}{8}x^3$

43. $P(x) = x^4 - 7x^2 + 5x + 5$; $Q(x) = x^4$

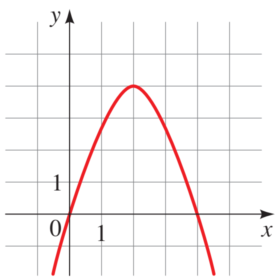
44. $P(x) = -x^5 + 2x^2 + x$; $Q(x) = -x^5$

45. $P(x) = x^{11} - 9x^9$; $Q(x) = x^{11}$

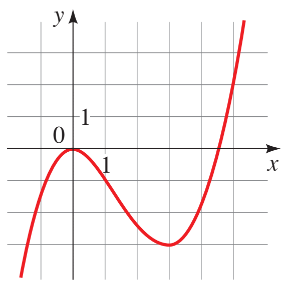
46. $P(x) = 2x^2 - x^{12}$; $Q(x) = -x^{12}$

47–50 ■ The graph of a polynomial function is given. From the graph, find **(a)** the x- and y-intercepts, and **(b)** the coordinates of all local extrema.

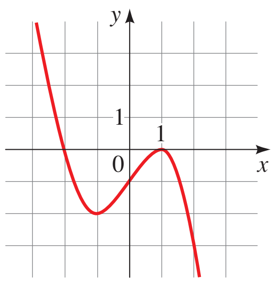
47. $P(x) = -x^2 + 4x$



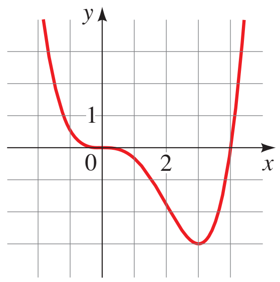
48. $P(x) = \frac{2}{9}x^3 - x^2$



49. $P(x) = -\frac{1}{2}x^3 + \frac{3}{2}x - 1$



50. $P(x) = \frac{1}{9}x^4 - \frac{4}{9}x^3$



51–58 ■ Graph the polynomial in the given viewing rectangle. Find the coordinates of all local extrema. State each answer rounded to two decimal places.

51. $y = -x^2 + 8x$, $[-4, 12]$ by $[-50, 30]$

52. $y = x^3 - 3x^2$, $[-2, 5]$ by $[-10, 10]$

53. $y = x^3 - 12x + 9$, $[-5, 5]$ by $[-30, 30]$

54. $y = 2x^3 - 3x^2 - 12x - 32$, $[-5, 5]$ by $[-60, 30]$

55. $y = x^4 + 4x^3$, $[-5, 5]$ by $[-30, 30]$

56. $y = x^4 - 18x^2 + 32$, $[-5, 5]$ by $[-100, 100]$

57. $y = 3x^5 - 5x^3 + 3$, $[-3, 3]$ by $[-5, 10]$

58. $y = x^5 - 5x^2 + 6$, $[-3, 3]$ by $[-5, 10]$

59–68 ■ Graph the polynomial and determine how many local maxima and minima it has.

59. $y = -2x^2 + 3x + 5$

60. $y = x^3 + 12x$

61. $y = x^3 - x^2 - x$

62. $y = 6x^3 + 3x + 1$

63. $y = x^4 - 5x^2 + 4$

64. $y = 1.2x^5 + 3.75x^4 - 7x^3 - 15x^2 + 18x$

65. $y = (x - 2)^5 + 32$

66. $y = (x^2 - 2)^3$

67. $y = x^8 - 3x^4 + x$

68. $y = \frac{1}{3}x^7 - 17x^2 + 7$

69–74 ■ Graph the family of polynomials in the same viewing rectangle, using the given values of c. Explain how changing the value of c affects the graph.

69. $P(x) = cx^3;\quad c = 1, 2, 5, \frac{1}{2}$

70. $P(x) = (x - c)^4;\quad c = -1, 0, 1, 2$

71. $P(x) = x^4 + c;\quad c = -1, 0, 1, 2$

72. $P(x) = x^3 + cx;\quad c = 2, 0, -2, -4$

73. $P(x) = x^4 - cx;\quad c = 0, 1, 8, 27$

74. $P(x) = x^c;\quad c = 1, 3, 5, 7$

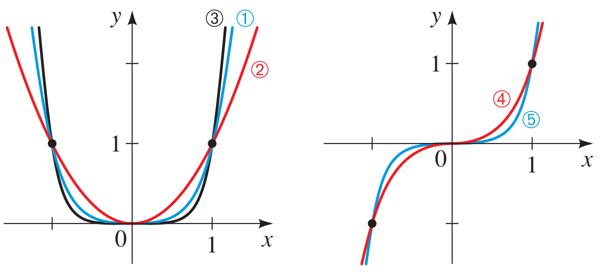
75. **(a)** On the same coordinate axes, sketch graphs (as accurately as possible) of the functions

$$y = x^3 - 2x^2 - x + 2 \qquad \text{and} \qquad y = -x^2 + 5x + 2$$

(b) On the basis of your sketch in part (a), at how many points do the two graphs appear to intersect?

(c) Find the coordinates of all intersection points.

76. Portions of the graphs of $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, and $y = x^6$ are plotted in the figures. Determine which function belongs to each graph.



77. Recall that a function f is *odd* if $f(-x) = -f(x)$ or *even* if $f(-x) = f(x)$ for all real x.

(a) Show that a polynomial $P(x)$ that contains only odd powers of x is an odd function.

(b) Show that a polynomial $P(x)$ that contains only even powers of x is an even function.

(c) Show that if a polynomial $P(x)$ contains both odd and even powers of x, then it is neither an odd nor an even function.

(d) Express the function

$$P(x) = x^5 + 6x^3 - x^2 - 2x + 5$$

as the sum of an odd function and an even function.

78. **(a)** Graph the function $P(x) = (x - 1)(x - 3)(x - 4)$ and find all local extrema, correct to the nearest tenth.

(b) Graph the function

$$Q(x) = (x - 1)(x - 3)(x - 4) + 5$$

and use your answers to part (a) to find all local extrema, correct to the nearest tenth.

79. **(a)** Graph the function $P(x) = (x - 2)(x - 4)(x - 5)$ and determine how many local extrema it has.

(b) If $a < b < c$, explain why the function

$$P(x) = (x - a)(x - b)(x - c)$$

must have two local extrema.

80. **(a)** How many x-intercepts and how many local extrema does the polynomial $P(x) = x^3 - 4x$ have?

(b) How many x-intercepts and how many local extrema does the polynomial $Q(x) = x^3 + 4x$ have?

(c) If $a > 0$, how many x-intercepts and how many local extrema does each of the polynomials $P(x) = x^3 - ax$ and $Q(x) = x^3 + ax$ have? Explain your answer.

APPLICATIONS

81. Market Research A market analyst working for a small-appliance manufacturer finds that if the firm produces and sells x blenders annually, the total profit (in dollars) is

$$P(x) = 8x + 0.3x^2 - 0.0013x^3 - 372$$

Graph the function P in an appropriate viewing rectangle and use the graph to answer the following questions.

(a) When just a few blenders are manufactured, the firm loses money (profit is negative). (For example, $P(10) = -263.3$, so the firm loses \$263.30 if it produces and sells only 10 blenders.) How many blenders must the firm produce to break even?

(b) Does profit increase indefinitely as more blenders are produced and sold? If not, what is the largest possible profit the firm could have?

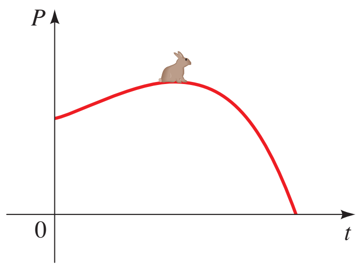
82. Population Change The rabbit population on a small island is observed to be given by the function

$$P(t) = 120t - 0.4t^4 + 1000$$

where t is the time (in months) since observations of the island began.

(a) When is the maximum population attained, and what is that maximum population?

(b) When does the rabbit population disappear from the island?

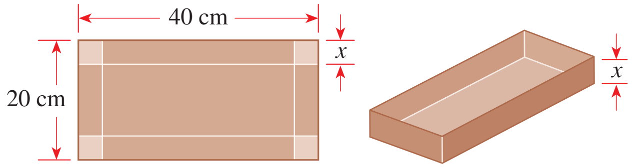


83. Volume of a Box An open box is to be constructed from a piece of cardboard 20 cm by 40 cm by cutting squares of side length x from each corner and folding up the sides, as shown in the figure.

(a) Express the volume V of the box as a function of x.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

(c) Draw a graph of the function V, and use it to estimate the maximum volume for such a box.

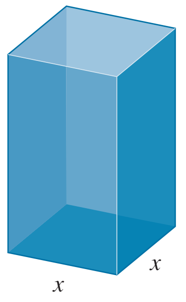


84. Volume of a Box A cardboard box has a square base, with each edge of the base having length x inches, as shown in the figure. The total length of all 12 edges of the box is 144 in.

(a) Show that the volume of the box is given by the function $V(x) = 2x^2(18 - x)$.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

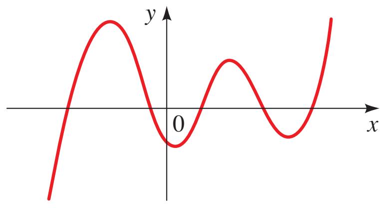
(c) Draw a graph of the function V and use it to estimate the maximum volume for such a box.



DISCOVERY ■ DISCUSSION ■ WRITING

85. Graphs of Large Powers Graph the functions $y = x^2$, $y = x^3$, $y = x^4$, and $y = x^5$, for $-1 \le x \le 1$, on the same coordinate axes. What do you think the graph of $y = x^{100}$ would look like on this same interval? What about $y = x^{101}$? Make a table of values to confirm your answers.

86. Maximum Number of Local Extrema What is the smallest possible degree that the polynomial whose graph is shown can have? Explain.



87. Possible Number of Local Extrema Is it possible for a third-degree polynomial to have exactly one local extremum? Can a fourth-degree polynomial have exactly two local extrema? How many local extrema can polynomials of third, fourth, fifth, and sixth degree have? (Think about the end behavior of such polynomials.) Now give an example of a polynomial that has six local extrema.

88. Impossible Situation? Is it possible for a polynomial to have two local maxima and no local minimum? Explain.

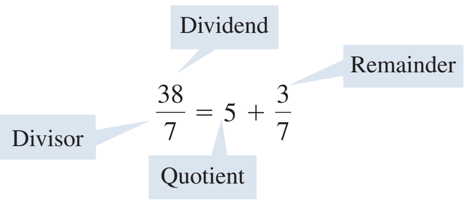
3.3 DIVIDING POLYNOMIALS

Long Division of Polynomials ▶ Synthetic Division ▶ The Remainder and Factor Theorems

So far in this chapter we have been studying polynomial functions *graphically*. In this section we begin to study polynomials *algebraically*. Most of our work will be concerned with factoring polynomials, and to factor, we need to know how to divide polynomials.

▼ Long Division of Polynomials

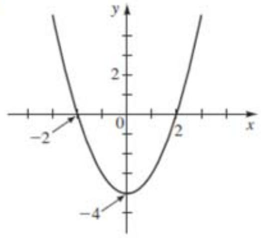
Dividing polynomials is much like the familiar process of dividing numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write



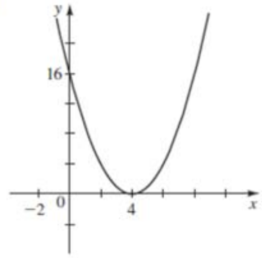
SECTION 3.2 ■ PAGE 243

1. II **2. (a)** (ii) **(b)** (iv) **3.** (a), (c) **4.** (a)

5. (a)



(b)



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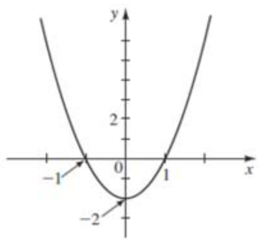

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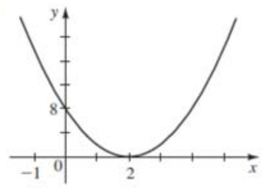
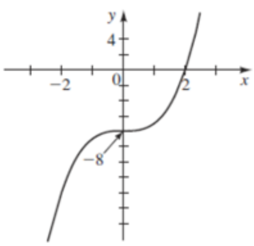


7. (a)



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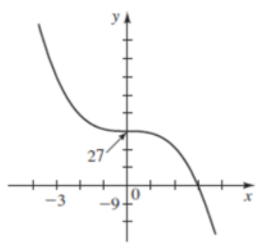

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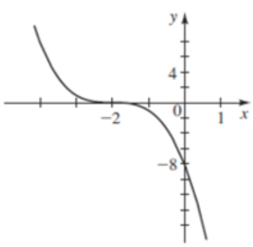

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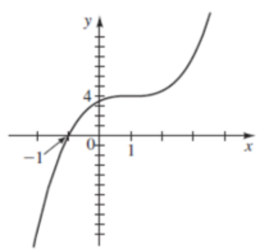
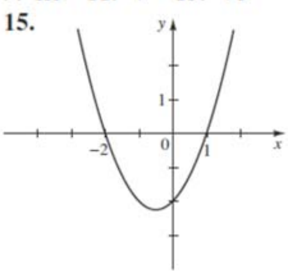


9. III **11.** V **13.** VI

15.



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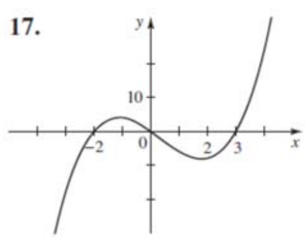

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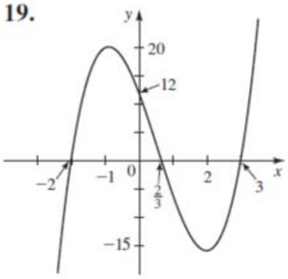

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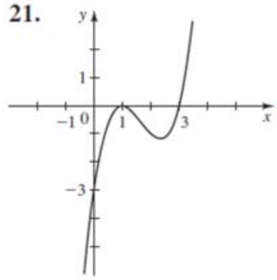

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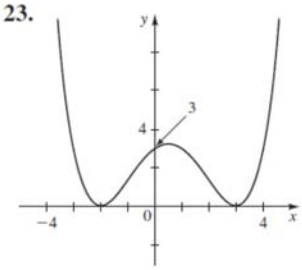

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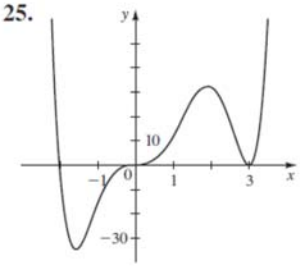
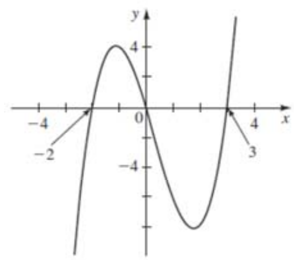
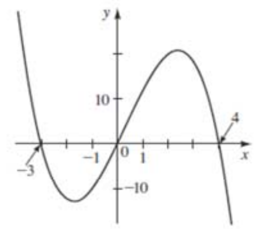


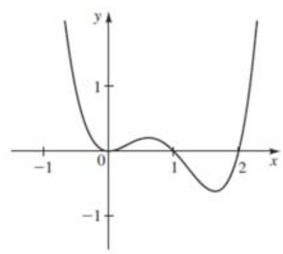
27. $P(x) = x(x + 2)(x - 3)$



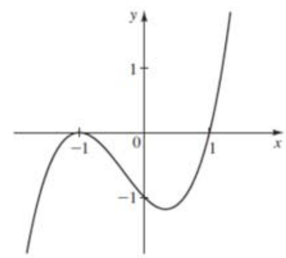
29. $P(x) = -x(x + 3)(x - 4)$



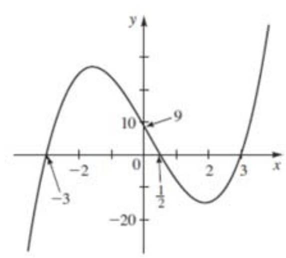
31. $P(x) = x^2(x - 1)(x - 2)$



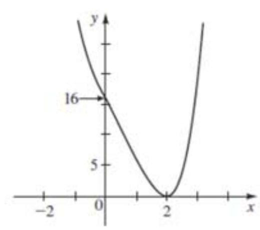
33. $P(x) = (x + 1)^2(x - 1)$



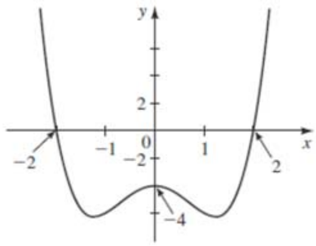
35. $P(x) = (2x - 1)(x + 3)(x - 3)$



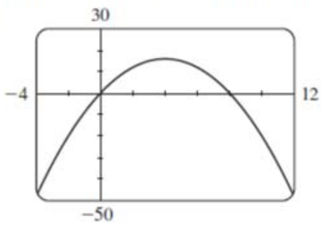
37. $P(x) = (x - 2)^2(x^2 + 2x + 4)$



39. $P(x) = (x^2 + 1)(x + 2)(x - 2)$

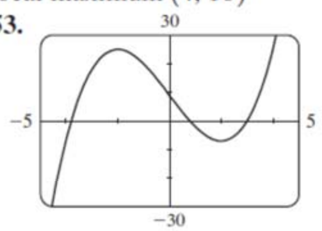


41. $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$
43. $y \to \infty$ as $x \to \pm\infty$
45. $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$
47. **(a)** x-intercepts 0, 4; y-intercept 0 **(b)** $(2, 4)$
49. **(a)** x-intercepts -2, 1; y-intercept -1 **(b)** $(-1, -2)$, $(1, 0)$
51.



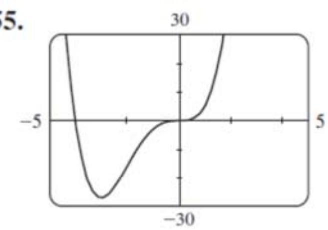
local maximum $(4, 16)$

53.



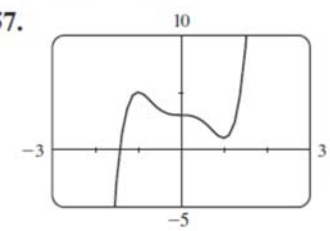
local maximum $(-2, 25)$,
local minimum $(2, -7)$

55.

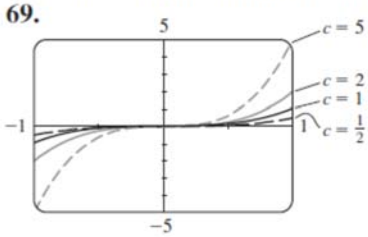


local minimum $(-3, -27)$

57.

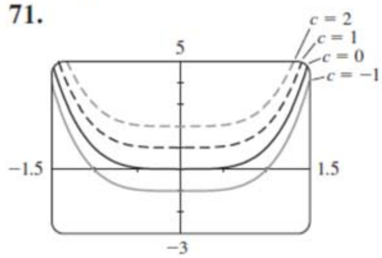


local maximum $(-1, 5)$,
local minimum $(1, 1)$
59. One local maximum, no local minimum
61. One local maximum, one local minimum
63. One local maximum, two local minima
65. No local extrema
67. One local maximum, two local minima
69.



Increasing the value of c stretches the graph vertically.

71.



Increasing the value of c moves the graph up.